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THE REVIEW: APPROACHES IN TEACHING CALCULUS

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ABSTRACT

Difficulty of learning calculus was faced by many students in higher secondary schools because the teaching was teacher centered and usually focused on explaining the notations and symbols used in calculus. Even after passing the examination, the students were not able to relate what they have learned in classroom in real life situations. With change of time the new approaches to teach calculus must be adopted so that what students learned in the classroom ac be also apply in real life situations. A few approaches used in teaching calculus that focus specifically on differential and integral calculus are reviewed: algebraic, graphical, contextual or realistic, and computer-assisted approaches.

Keywords: Calculus, Algebraic, Graphical, Realistic, Computer-Assisted Approaches

INTRODUCTION

Calculus teaching has been revolutionized for the last three decades due to the contributions of a number of mathematics educators. The technology-driven changes inspired a major university-centered “Calculus Reform Movement” in the late 1980s and early 1990s [1]. The three events, “Towards a Lean and Lively Calculus” conference in 1986, “Calculus for a New Century” symposium in 1987, and “Priming the Calculus Pump” in 1990, signaled the beginning of calculus reform [2] to make calculus lean and lively, relevant to applications by including contemporary mathematics that reflected new technology. The themes of calculus reform movement include: involving students in doing mathematics instead of lecturing at them; stressing conceptual understanding rather than only computation; developing meaningful problem-solving abilities; exploring patterns and relationships instead of just memorizing formulas; becoming engaged in open-ended, discovery-type problems rather than doing routine, closed-ended exercises; and approaching mathematics as a live exploratory subject, not merely as a description of past works

There might be several approaches, but only a few approaches used in teaching calculus that focus specifically on differential and integral calculus are reviewed: algebraic, graphical, contextual or realistic, and computer-assisted approaches.

ALGEBRAIC APPROACH

An algebraic approach is most closely related to the traditional approach of teaching calculus. Elk [3] utilized a lecture-based format representing a logical extension of algebra to transition into calculus courses. He proposed the basic algebraic ideas imbedded in the concepts of derivative and integral. He cited an example of three situations that involve division by zero: 0/*a*, *a*/0, and 0/0 where '*a*' is any number not equal to zero. He emphasized that differentiation gives meaning to the third case (0/0) under the appropriate conditions. He introduced the idea of the changes in both the *y*-coordinate and the *x*-coordinate approaching zero. As the distance between the points decreases, the two respective components of the distance decrease and become very small. Thus, the change in the *y*-coordinate over the change in the *x*-coordinate would cause both the numerator and the denominator to be zero, resulting in the 0/0 situation.

Consider the function $f(x)=\dfrac{(x^2-4)}{(x-2)}$. At x = 2, the function reduced to 0/0, and at this point, the function is not defined. In order for the function to be defined at all points, an additional criteria needs to be added; for example, if $f(x)$ = 4 were added, it would produce a continuous function for all values of x. Furthermore, any function $f(2) = b$ where b is any number not equal to 4 would produce a discontinuous function. Using this information, Elk presented an algorithm to compute the derivative of the given function:

i). From the given function $y = f(x)$, form $f(x+\Delta x)$.

ii). Subtract $f(x)$ from $f(x+\Delta x)$, a difference we call ΔY.

iii). Divide ΔY by ΔX and simplify the algebra.

iv). Take the limit as $\Delta X \to 0$.

The algorithm results in the equation $y' = \lim\limits_{X\to 0}\dfrac{\Delta y}{\Delta x}$, or $y' = \lim\limits_{X\to 0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$, the definition of the derivative. The key to giving meaning to 0/0 is taking the limit at the end of the algorithm rather than during one of the earlier steps. By doing this, an initially undefined computation can bring about information. Similarly, Elk proposed that integration gives meaning to multiplying infinity by zero. The process of dividing a figure into small pieces to calculate the sum of the areas and repeating the calculation by sub-dividing further until the pieces are infinitesimal yields the desired area as the limit. This is equivalent to multiplying infinity by 0. He stressed the importance of noting that this multiplication oftentimes yields a finite product, contrasting the common notions that zero multiplied by anything is zero and that infinity multiplied by anything is infinity. Thus, not only does integration give meaning to infinity times zero, but that meaning is oftentimes finite. This can be seen given the standard definition of an integral: $\int_a^b f(x) = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$; the limit is the sum of an infinite number of zero terms since the limit of each Δx_i is zero.

Although the algebraic approach encourages a better conceptual understanding of the ideas of differentiation and integration, utilizing the algebraic approach may limit the capability of students to extend their knowledge beyond the definitions of derivative and integral to problems concerning real world applications. Elk also mentioned that integration often seems to give a finite value to the meaning of infinity times zero, it does not explain how that value is reached other than the inclusion of the limit in the standard definition. Furthermore, the definition of the derivative is still structured procedurally, and a concern may be that the step-by-step instructions do not adequately deviate away from lecture-based instruction.

GRAPHICAL APPROACH

The advantages of using computer technology were beginning to surface. Tall [4] saw the potential of computers and presented a graphical approach to teach integration and the fundamental theorem. He mentioned the difficulty of going beyond simple examples when using the algebraic method of summation, and the tediousness and obscurity of interpretation that occurred when using a calculator. For these reasons, he presented a process that involves progressing from paper to calculator and then to the graphical interface of a computer in order to obtain insight into the concepts of integration.

His main approach to teaching integration was to have students infer possible functions that described the behavior observed through the approach taken to solve the integral, be it through the calculator or computer software. He illustrated the difficulty in conjecturing the formula for the area under a graph over an interval using the averages of upper and lower sums (the trapezium rule) since the numerical work begins to get oppressive unless a computer is available. As a solution to this, he mentioned the study of Neill and Shuard [5] where they cunningly involved the entire class to produce a table of areas under the graphs of x^n from 0 to 1 using the trapezium rule with 10 strips with the intention of leading students to conjecture that the area under $f(x) = x^n$ from 0 to 1 is $\frac{1}{n+1}$. From the table, students could see the common fractions $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, etc.

Tall proposed using the information obtained from partial sums to conjecture the areas of functions more directly than the results of area calculations. By plotting the successive accumulations of areas of the function $f(x) = x^2$ (see Figure 1.1) as separate points that represented the areas under the graph from *x* = 0 to the current *x*-coordinate using the mid-ordinate rule, the area graph looked like a higher power of *x*. Students then conjectured that it looked about like kx^3 for some constant *k*. Since the area graph and the graph of x^2 crossed at *x* = 3, substituting the value of *x* yielded the value of $k = \frac{1}{3}$.

Tall's idea behind the graphical approach was to concentrate on the ideas rather than the technicalities. For example, as explained above, students were meant to become aware of two ideas: approximations of the area grew closer to the true area when smaller strips were taken, and the results of the graphs displayed the patterns. However, it should be noted that he expected students to be able to calculate approximations numerically before graduating to software that eliminated the need for calculations by hand. With that knowledge base, students could use the software to explore cases such as negative areas and develop concepts such as the fundamental theorem graphically. There are two kinds of negative areas: one resulting from areas under the *x*-axis and another resulting from integrating toward the left of the lower limit of the integral. The latter involves taking steps in the negative direction, which was rarely done in traditional courses, and teachers often said that it was too difficult for their students. Graphically, students could understand that a negative step and a positive ordinate resulted in a negative product, and that a negative step and a negative ordinate resulted in a positive product. Tall presented a computer simulation that went beyond a static picture to see the negative steps and sense the growth of the area as the picture developed. By investigating the negative steps, students could uncover all four sign combinations.

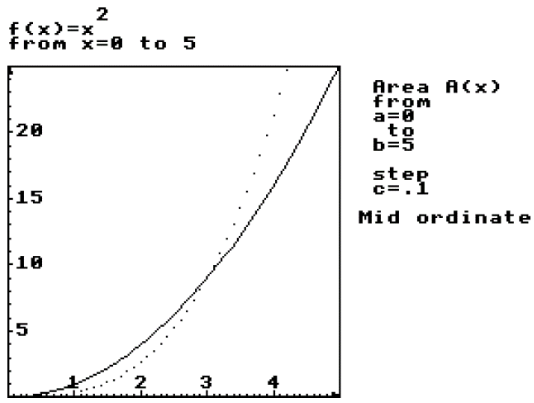

Figure 1.1 The cumulative area under $f(x) = x^2$ from 0 to 5.

After observing that smaller strips resulted in area approximations that came closer to the true area and understanding the signs, Tall believed that the students now had the background knowledge necessary to find the area between a graph and the *x*-axis from a fixed point *a* to a variable *x* as a function of *x*. By plotting the cumulative area function for the graph $f(x)=x$, starting at the origin and moving right, and then restarting at the origin and moving left, the graph took on the recognizable shape of a quadratic function kx^2 (Figure 1.2). Then by noting that when $x=2$, $y=2$ the students could conjecture that the area of the function was $\frac{x^2}{2}$. The same process can be applied to $f(x)=x^2$ and the general pattern of the area under $f(x)=x^n$ from 0 to *x* can be thought of as $f(x)=\frac{x^{n+1}}{n+1}$.

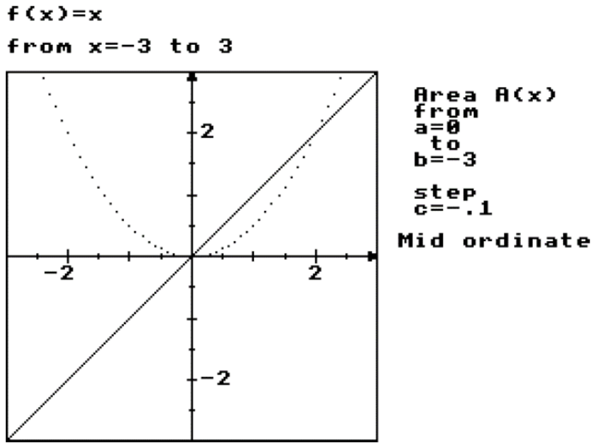

Figure 1.2 The area under $f(x)=x$ calculated from $x=0$.

Gravemeijer and Doorman [6] noted the characteristic dynamical aspect of Tall's [7] graphical approach to teaching the introductory calculus concepts. When teaching derivatives, the graphs showed how the dependent variable (*y* or *f*(*x*)) changed when the independent value *x* changed at a constant rate. Students then observed the changes in the dependent variable and the rate of these changes and began to develop an intuitive idea of these changes in terms of increase, decrease, and gradient.

Gravemeijer and Doorman also demonstrated that focusing on the gradient of the graph leaved the idea of the derivative as a measure of the rate of change as implicit. The difficulty lay in the transition from visual imagery discussion to formal mathematical reasoning. Hence, they concluded that students interpreted a definition that was based on visual imagery as a description of the picture, instead of a mathematical definition that could be used for formal reasoning.

CONTEXTUAL APPROACH

James Kaput [8] developed an approach to teaching calculus that related mathematical symbols and graphs to everyday realities. Kaput devoted his time describing an early version of the video system *MathCars* [9] that simulated driving a car through interactive technology [8]. The idea behind the software was to map the phenomenologically rich experience of motion in a vehicle (sights and sounds) onto coordinate graphical and other mathematical notations by controlling aspects such as time, distance, and velocity, which were visible as visual representations or graphs [10]. Gravemeijer and Doorman [6] used the metaphor of the software linking “the gap between the island of formal mathematics and the mainland of real human experience”. In the case of *MathCars*, the software linked the everyday experience of motion in a vehicle with formal graphical representations while Elk’s [3] approach was built upon knowledge learned in algebra.

The Dutch approach called Realistic Mathematics Education (RME) uses contextual problems as “models of” reality, which gradually progress to “models for” mathematical reasoning [6]. Instead of a procedural [3], [10] or an entirely graphical approach [4], [12], [13], which Gravemeijer and Doorman believed that it may leave students unable to understand the whole picture or unable to connect concepts to formal mathematics, RME suggests that teachers employ guided re-invention, often by considering the historical origins of the concepts. The idea is to keep the gap between where the students are and what is being introduced as small as possible by designing a hypothetical learning trajectory for the students to be able to reinvent formal mathematics. While Kaput [8] used a historically based approach as well, the perspective shift from considering notation and human experience as distinct to viewing them as a development from one to the next is not comparing a ready-made system with real world experiences but rather a reinventing process of the system.

The process of guided re-invention emphasizes the character of the learning process over the invention. The idea is for the students to develop their own private knowledge and understanding individually, with guidance, and no answers from the teacher. The idea of progressive mathematization that Gravemeijer and Doorman presented stems from a combination of Treffers’s idea of horizontal and vertical mathematization [14] Horizontal mathematization is the process of describing a context problem in mathematical terms. Vertical mathematization is mathematizing one’s own mathematical activity, reaching higher levels of mathematics through the process.

By using contextual problems, students develop informal, context-specific solution strategies that are meant to guide them to generalizations. These generalizations become models for mathematical reasoning. In the case of RME, contextual problems are defined as problem situations that are experientially real to the students [6] As an overview, in order to develop calculus concepts, contextual problems consisting of modeling problems about velocity and distance were employed. Using this method to understand integrals, students initially developed discrete approximations of a function denoting varying velocities and distances. The resulting inscribed discrete graphs later became continuous and the model for formal mathematical reasoning. The act of summing the discrete intervals made on the graph or noting the differences in the increments were the reifying processes, while the integral and derivative became the mathematical objects. As the students worked through the problems, they would likely experience a nonlinear learning process that would end with a result between a process and an object. The goal for the teacher was to emphasize that the underlying process was an integral part of the mathematical object that was developed [6]

The following presented the reification process of the integral and then the derivative. Instead of simply being told that the area of the graph coincided with the total distance covered over a period of time, students developed the ability to model and conceptualize motion through representations and approximations. In order to accomplish this, the students were first presented with the story of Galileo. He presumed that a free-falling object would move with a constantly increasing velocity. Students were asked to graph the discrete approximation of that motion and then asked to discover the distance covered by the object (see Figure 1.3) [6]

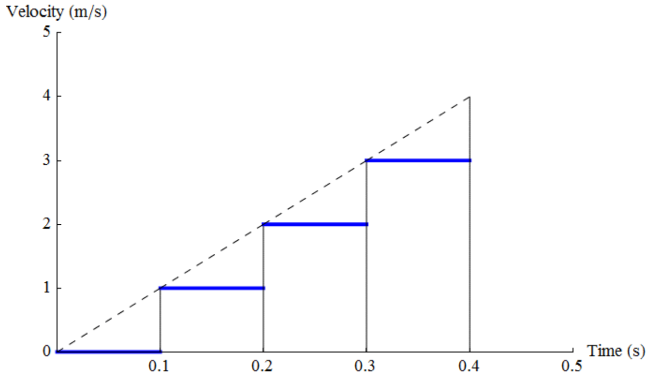

Figure 1.3 A discrete approximation of a constantly changing velocity.

Students solved these discrete approximations and were then expected to connect the area of the discrete graph with the area of a continuous graph, $s(t) = \frac{t \times 10t}{2} = 5t^2$, which is the area of a triangle (with base t and height $10t$). From this, they discovered the quadratic relationship between time and distance, which was what Galileo used to test the above-mentioned hypothesis. The next step was to progressively rediscover the differential calculus. This was done by determining velocities from the distance-time graphs and formulas. As students worked through examples, the model they had developed for reasoning through the problems would begin to function as a model for reasoning arbitrary function as well as standard algebraic functions. At this point, the teacher should shift from an everyday-life contextual problem to a focus on mathematical concepts and relations. This shift could be possible for the students when they were able to develop a framework that enabled them to view the problems mathematically.

Kaput [8] also suggested employing a curriculum that introduced the ideas of calculus (variable rates of changing quantities, the accumulation of those quantities, the connections between rates and accumulations, and approximations) in early grades. The idea was to build up concepts gradually in the mathematics found in calculus. In order to develop curriculum, he considered the implications of history and desired to look closely at the origins of the major ideas of calculus for clues regarding how calculus might be regarded as a web of ideas that should be approached gradually, from elementary school onward, in a coherent school mathematics curriculum. Overall, Kaput mentioned that mathematicians in the past developed the ideas gradually based on real world experiences. With this in mind, Kaput suggested a three-step approach for developing the ideas of derivatives and integrals.

Dubinsky [10] outlined the three steps mentioned by Kaput. Firstly, at an early age, students should be able to represent quantities such as temperature, velocity, acceleration, time, and distance as geometric objects (i.e. lines and rectangles). Secondly, teachers incorporated video technology such as *MathCars* in order to help students conceptualize continuous phenomena. Lastly, students discussed and moved towards firm, logical foundations of the principles of calculus.

This approach aims for students to develop a conceptual understanding of the concept smoothly as they are guided from contextual problems to a focus on mathematical concepts and relations. However, the formal definition, containing a limit and based on summing the areas of infinitely many rectangles, is not introduced; only the geometric area under a curve is discussed. Although the jump from true area to approximated area may not be a substantial one, it is to be considered if students are expected to be able to understand the formal definition of an integral.

COMPUTER-ASSISTED APPROACH

Lang [15] developed computer-assisted calculus instruction (CAI) in China. Although technology has developed significantly since the article was written, it presented the idea of implementing laboratory courses to foster the scientific exploratory spirit of the undergraduate calculus students at an engineering school. Lang emphasized the role of the computer as a tool, not a replacement for brainpower, and noted that good problems should encourage students to think deeply. The goal was for the laboratory to be continuously in development, and Lang encourages teacher collaboration to develop good problems.

Since it was assumed that the students should be taught the formulas for integrals and derivatives in a more traditional setting in the classroom, the laboratory was the time for students to be able to understand concepts and pursue novelty in mathematics [15] The idea was to shift the students away from memorizing the proof of a theorem to truly understanding the theorem. Therefore, the laboratory was the time for students to find 'explanations' instead of 'proofs' for the definitions they had learned. Lang illustrated that good explanations included important ideas in proof, and thus a student's understanding of a mathematical theorem was not necessarily weakened by a shift towards a concentration on explanation over proof.

The following was an example of a computer experiment to develop the understanding of the relationship between functional increment and differential [15]

For $f(x)=\sqrt{x}$ when $x_0=64$, $\Delta x=-1$, find the true increment and its differential approximation, make a comparison, and calculate the relative error when using the approximation.

For $g(x)=x^{100}$ when $x_0=1$, calculate the increment and differential approximation, make a comparison and calculate the errors at $\Delta x=0.03$, $\Delta x=0.003$ and $\Delta x=0.0003$ respectively.

By calculating and comparing increments and their differential approximations of various functions that resulted from different increments for different values of the independent variable, students could discover the accuracy of the approximation. Students noted that the differential approximation became a good approximation of the function increment when the sizes of the increment of the independent variable were smaller.

The idea of the laboratory that Lang mentioned was to shift the students away from the memorization of definitions and proofs and towards an ability to explain these definitions. Although only a couple examples of questions posed to students in the laboratory were presented in the article, the questions seemed to focus on either exploring the limits of the procedures used to calculate integrals and derivatives or identifying patterns in relationships. While this knowledge was helpful in understanding the processes, it did not suggest a change in the way the definitions of integrals and derivatives were presented to the students in the lecture portion of the classroom or hinted towards any contextual problems or applications.

The four aforementioned approaches were an effort to challenge traditional calculus instruction and to aid students in developing a better conceptual understanding of the meanings of differentiation and integration. Although each approach presented various strategies to help students develop the concepts, each strived to move beyond procedural fluency. The differing methods incorporated students' prior knowledge, historical influence, real-world contexts, graphical images, and computer-enhanced computations.

CONCLUSION

The teaching approaches mentioned above are to aid students in developing a conceptual understanding of the meanings of differentiation and integration. Although each approach presented various strategies of the concepts to students, each strived to move beyond procedural fluency. The differing methods incorporated students' prior knowledge, historical influence, real-world contexts, graphical imagery, and computer-enhanced computations.

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